ASSESSING THE TIME DEPENDENCE OF MULTIVARIATE EXTREMES

FOR HEAVY RAINFALL MODELING

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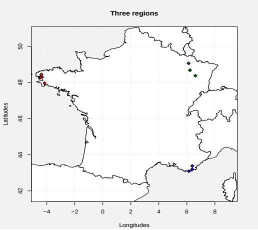

Figure 1: Map of France locating three different climate regions for a case study of fall daily rainfall amounts from 1976 to 2015.

- Heavy rainfall modeling is needed to design prevention plans against disasters. For this reason, water cycle research has a **high environmental, societal, and economic impact**.

- Heavy rainfall modeling is critical for **risk evaluation** of natural hazards like flooding, debris flows, and landslides.

- Rainfall amounts reach high-intensity levels frequently. This feature is modeled by hydrologist with **heavy-tailed distributions**.

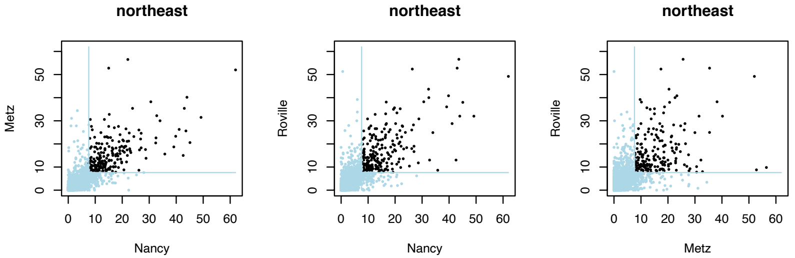

Figure 2: Scatter plot of fall daily rainfall amounts from 1976 to 2015 for three neighboring stations in the northwest of France.

$$|\mathbf{X}_t| = \max_{j=1,2,3} X_{t,j}.$$

In black, simultaneous exceedances of the 95-th empirical quantile of $(|\mathbf{X}_t|)_{t=1,\ldots,n}$.

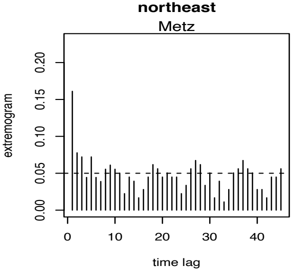

Figure 3: Empirical temporal extremogram $\widehat{\chi_t}$:

$$\chi_t \quad = \quad \lim_{x \to +\infty} \mathbb{P}(|\mathbf{X}_t| > x \,|\, |\mathbf{X}_0| > x)\,,$$

of the 95-th order statistic of fall daily rainfall levels recorded at Brest from 1976 to 2015. As a baseline, the extremogram takes the dotted line pointed value at independent time lags.

Goals

1- To model accurately space and time dependencies, we **present a theoretical framework** for studying multivariate stationary heavy-tailed time series.

2- For our application, it is common to record a big storm simultaneously at close stations. We aim to **propose new statistical methodologies** to aggregate thoughtfully the spatiotemporal extreme observations of the underlying event.

Outline

Introduction

Extremal ℓ^p-blocks

p-cluster theory

Examples

Statistical methods

p-cluster inference

Numerical experiments

Introduction

We consider

- $(\mathbf{X}_t)$ stationary time series in $(\mathbb{R}^d, |\cdot|)$.
- $\|(\mathbf{x}_t)\|_p = (\sum_{t\in\mathbb{Z}} |\mathbf{x}_t|^p)^{1/p}$, for $p \in (0,\infty)$, the supremum norm $\|(\mathbf{x}_t)\|_\infty$ for $p = \infty$.

$$\|(\mathbf{x}_t)\|_\infty \;\leq\; \|(\mathbf{x}_t)\|_p \;\leq\; \|(\mathbf{x}_t)\|_q, \quad p > q.$$

- (ℓ^p, d_p) sequential metric space.
- $(\mathbf{x}_t)_{t=a,\dots,b} = \mathbf{x}_{[a,b]}$.

Consider $(\mathbf{X}_t)$ satisfies $\mathbf{RV}_\alpha$ if for all $b \geq 1$, $\mathbf{X}_{[1,b]} \in \mathbb{R}^{d\times b}$ is **multivariate regularly varying**, i.e.,

$$\mathbb{P}\Big(\|\mathbf{X}_{[1,b]}\|_p \;>\; y\,x\,,\; \frac{\mathbf{X}_{[1,b]}}{\|\mathbf{X}_{[1,b]}\|_p} \;\in \cdot \;\Big|\; \|\mathbf{X}_{[1,b]}\|_p \;>\; x\Big)$$
$$\xrightarrow{w} \quad y^{-\alpha}\, \mathbb{P}(\mathbf{Q}^{(p,b)} \;\in \cdot\;)\,, \quad x \to +\infty, \quad y > 1\,,$$

such that $\mathbf{Q}^{(p,b)} \in \mathbb{R}^{d\times b}$, and $\|\mathbf{Q}^{(p,b)}\|_p = 1$ a.s.

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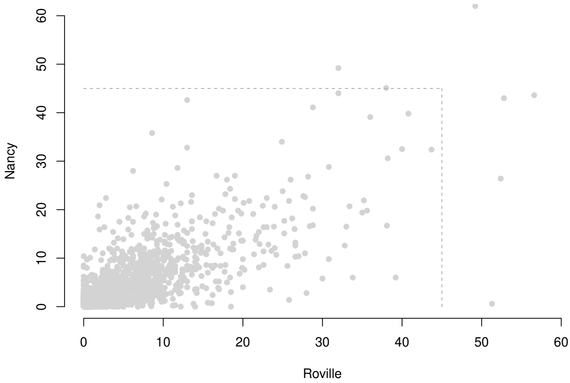

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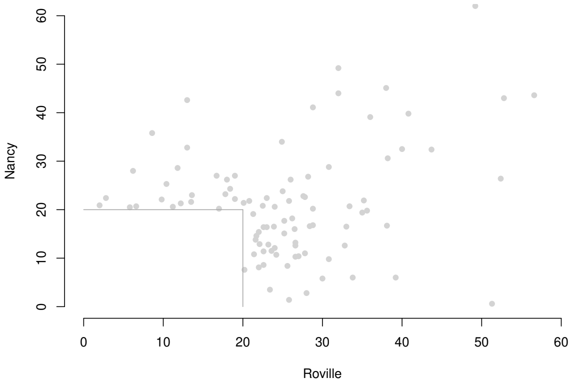

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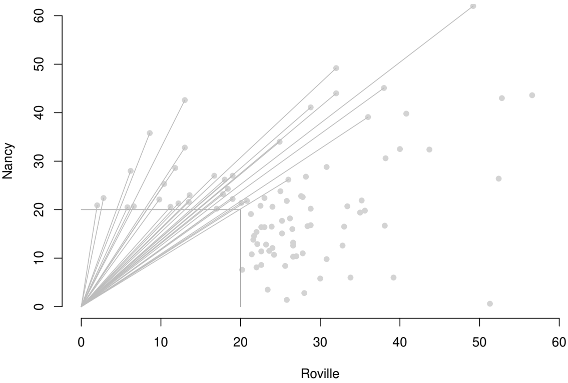

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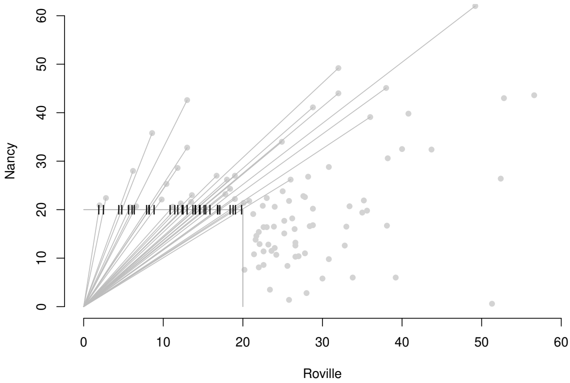

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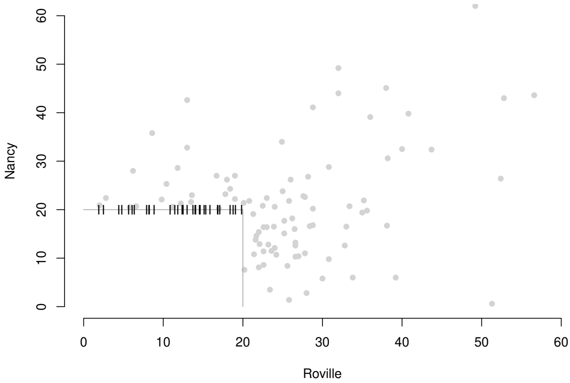

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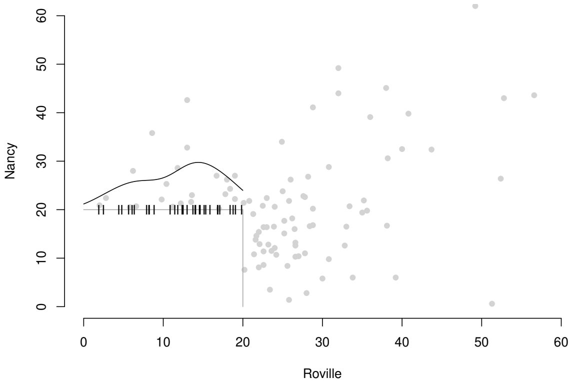

Modeling perspectives

Let $(\mathbf{X}_t)$ be a stationary time series satisfying $\mathbf{RV}_\alpha$.

$$\begin{aligned} &\mathbf{X}_{[1,n]}/x \\ &\quad = \quad (\,\mathbf{X}_{[1,b]}/x\,,\ \mathbf{X}_{[b+1:2\,b]}/x\,,\ \ldots\ ,\ \mathbf{X}_{[n-b+1:n]}/x\,), \\ &\qquad\quad x \longrightarrow +\infty. \end{aligned}$$

- Condition $\mathbf{RV}_\alpha$ models intra-blocks dependence.
- There might still be inter-blocks dependence!

Let $(\mathbf{X}_t)$ be a stationary time series satisfying $\mathbf{RV}_\alpha$.

$$\begin{aligned} &\mathbf{X}_{[1,n]}/x_{b_n} \\ &\quad = \quad (\,\mathbf{X}_{[1,b_n]}/x_{b_n}\ ,\ \mathbf{X}_{[b_n+1:2\,b_n]}/x_{b_n}\ ,\ \ldots\ ,\ \mathbf{X}_{[n-b_n+1:n]}/x_{b_n}\,), \\ &\qquad n \longrightarrow +\infty. \end{aligned}$$

- Disjoint blocks become asymptotically independent under mixing conditions.
- Our goal is to model the extremal features of blocks $\mathbf{X}_{[1,b_n]}$ in (ℓ^p, d_p) such that $\|\mathbf{X}_{[1,b_n]}\|_p > x_{b_n}$.
- We focus on large deviations of blocks, i.e.,

$$\mathbb{P}(\textstyle\sum_{t=1}^{b_n} |\mathbf{X}_t|^p > x_{b_n}^p) \;=\; \mathbb{P}(\|\mathbf{X}_{[1,b_n]}\|_p > x_{b_n}) \;\to\; 0.$$

1. **Heavy-tailed time series**

   Blocks of maxima[1] $p = \infty$

   Extremal index[2] $\theta_{|\mathbf{X}|} \in (0, 1]$,

$$(\mathbb{P}(|\mathbf{X}_1| \le x a_n))^n \quad \to \quad G(x), \qquad \mathbb{P}(\|\mathbf{X}_{[1,n]}\|_\infty \le x\, a_n) \quad \to \quad (G(x))^{\theta_{|\mathbf{X}|}}.$$

2. **Sums of regularly varying increments $p < \infty$** [3]

   Large deviations of sums

   Central limit theory

$$(\textstyle\sum_{t=1}^n \mathbf{X}_t - b_n)/a_n \quad \xrightarrow{d} \quad \xi, \qquad n\mathbb{P}(|\mathbf{X}_1| > a_n) \to 1.$$

---

[1]Davis & Hsing (1995), Basrak & Segers (2009), Basrak *et al.* (2018), Janßen (2019), Kulik & Soulier (2020).

[2]Leadbetter (1983), Leadbetter *et al.* (1983), C.Y. Robert (2008).

[3]Davis & Hsing (1995), Jakubowski (1997), Bartiewicz *et al.* (2011), Basrak *et al.* (2012), Mikosch & Wintenberger (2013), Basrak *et al.* (2018).

Extremal ℓ^p-blocks

Theorem 1

Assume $(\mathbf{X}_t)$ satisfies $\mathbf{RV}_\alpha$, and (x_n) satisfies $\mathbf{AC}(x_n)$, $\mathbf{CS}_p(x_n)$, and $\mathbb{P}(\|\mathbf{X}_{[1,b_n]}\|_p > x_{b_n}) \to 0$. Then,

$$\mathbb{P}(\|\mathbf{X}_{[1,b_n]}\|_p > x_{b_n})/(b_n\mathbb{P}(|\mathbf{X}_0| > x_{b_n})) \quad \to \quad c(p),$$

$\theta_{|\mathbf{X}|} = c(\infty) \leq c(p) \leq c(\alpha) = 1$, for $p \in (\alpha, \infty)$.

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$$\mathbb{P}\Big(\|\mathbf{X}_{[1,b_n]}\|_p > y\, x_{b_n}, \frac{\mathbf{X}_{[1,b_n]}}{\|\mathbf{X}_{[1,b_n]}\|_p} \in \cdot \mid \|\mathbf{X}_{[1,b_n]}\|_p > x_{b_n} \Big)$$
$$\xrightarrow{d} \quad y^{-\alpha}\, \mathbb{P}(\mathbf{Q}^{(p)} \in \cdot),$$

such that $\mathbf{Q}^{(p)} \in (\ell^p, d_p)$, $\|\mathbf{Q}^{(p)}\|_p = 1$ a.s., and convergence holds for a family of shift-invariant ℓ^p-continuity sets.

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Let $(\mathbf{X}_t)$ be a stationary time series in $(\mathbb{R}^d, |\cdot|)$ satisfying $\mathbf{RV}_\alpha$ such that

- $\mathbf{AC}(x_n)$: for all $\epsilon > 0$,

$$\lim_{k\to+\infty} \limsup_{n\to+\infty} \mathbb{P}(\|\mathbf{X}_{[k,n]}\|_\infty > \epsilon\, x_n \,|\, |\mathbf{X}_0| > \epsilon\, x_n) \;=\; 0.$$

[4]Davis & Hsing (1995), Jakubowski (1997), Bartkiewicz *et al.* (2011) for stable limit theorems.
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- $\mathbf{CS}_p(x_n)$[4]: for all $\epsilon, \delta > 0$.

$$\lim_{\epsilon\downarrow 0} \limsup_{n\to+\infty} \frac{\mathbb{P}(\sum_{t=1}^n |\mathbf{X}_t|^p \mathbb{1}(|\mathbf{X}_t| \le \epsilon\, x_n) > \delta\, x_n^p)}{n\mathbb{P}(|\mathbf{X}_0| > x_n)} = 0.$$

This holds for all $p > \alpha$ under $\mathbf{RV}_\alpha$ by Karamata's theory.

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Assumptions

Let $(\mathbf{X}_t)$ be a stationary time series in $(\mathbb{R}^d, |\cdot|)$ satisfying $\mathbf{RV}_\alpha$ such that

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- $\mathbf{RV}_\alpha$[5]: there exists $(\mathbf{\Theta}_t)$ such that $|\mathbf{\Theta}_0| = 1$ a.s., and for all $h \ge 0$, $y > 1$,

$$\lim_{x\to+\infty} \mathbb{P}(|\mathbf{X}_0| > y\, x, \frac{\mathbf{X}_{[-h,h]}}{|\mathbf{X}_0|} \in \cdot \,|\, |\mathbf{X}_0| > x) \;=\; y^{-\alpha}\, \mathbb{P}(\mathbf{\Theta}_{[-h,h]} \in \cdot).$$

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p-CLUSTER THEORY

Proposition 1

Under **AC** the constant of large deviations satisfies

$$c(p) \quad = \quad \mathbb{E}[\|\boldsymbol{\Theta}\|_p^\alpha / \|\boldsymbol{\Theta}\|_\alpha^\alpha].$$

Moreover, if $c(p) < \infty$ and **AC**, $\mathbf{CS}_p$ hold then the p-cluster process admits the representation in (ℓ^p, d_p).

$$\mathbb{P}(\mathbf{Q}^{(p)} \in \cdot) \quad = \quad c(p)^{-1}\mathbb{E}\big[\|\boldsymbol{\Theta}\|_p^\alpha / \|\boldsymbol{\Theta}\|_\alpha^\alpha \, 1\big(\boldsymbol{\Theta}/\|\boldsymbol{\Theta}\|_p \in \cdot\big)\big].$$

Recall[6] $\|\boldsymbol{\Theta}\|_\alpha < \infty$ as soon as $|\boldsymbol{\Theta}_t| \to 0$ as $t \to \infty$, which holds under **AC**.

[6] Janßen (2019)

Corollary 1

If $(\mathbf{X}_t)$ satisfies $\mathbf{RV}_\alpha$, **AC**, $\mathbf{CS}_\alpha$, then $c(\alpha) = 1$ and $\mathbf{Q}^{(\alpha)} = \mathbf{\Theta}/\|\mathbf{\Theta}\|_\alpha$ a.s.

The normalizing constant $c(p) = \mathbb{E}[\|\mathbf{\Theta}/\|\mathbf{\Theta}\|_\alpha\|_p^\alpha]$ depends on the temporal dependence except when $p = \alpha$ and

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The normalizing constant $c(p) = \mathbb{E}[\|\mathbf{\Theta}/\|\mathbf{\Theta}\|_\alpha\|_p^\alpha]$ depends on the temporal dependence except when $p = \alpha$ and

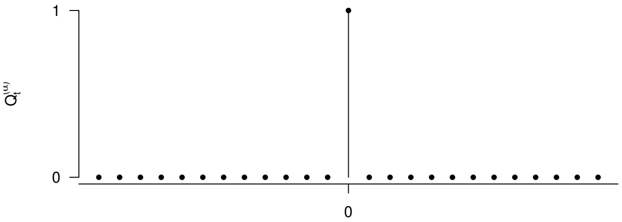
$$\mathbb{P}(\|\mathbf{X}_{[1,b_n]}\|_\alpha > x_{b_n}) \quad \sim \quad b_n \mathbb{P}(|\mathbf{X}_0| > x_{b_n}).$$

The ℓ^α–blocks exceed high levels at a constant rate mindless of the time dependencies. For this reason, inference based on ℓ^α-blocks is robust to handle time-dependencies!

Examples

- $(\mathbf{X}_t)$ **i.i.d.**, $\mathbf{X}_1$ satisfies $\mathbf{RV}_\alpha$,

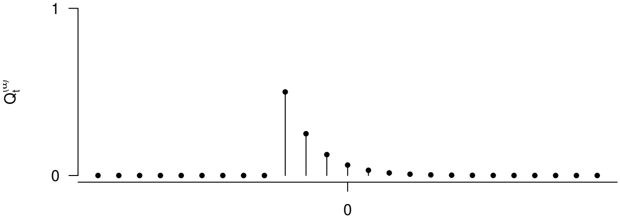
$$\mathbf{Q}_t^{(\alpha)} \quad = \quad \mathbf{\Theta}_t \quad = \quad \mathbb{1}(t = 0)\,\mathbf{\Theta}_0 .$$



- (X_t) a stationary **AR(1)**, $X_t = \varphi X_{t-1} + Z_t$ with $\varphi \in (0,1)$, and (Z_t) i.i.d. satisfying $\mathbf{RV}_\alpha$,

$$Q_t^{(\alpha)} \quad = \quad \Theta_t / \|\Theta\|_\alpha \quad = \quad \varphi^t \Theta_0^Z \ \mathbb{1}(J + t \geq 0)\, (1 - \varphi^\alpha)^{1/\alpha},$$

J independent of Θ_0^Z, $\mathbb{P}(J = j) = (1 - \varphi^\alpha)\varphi^{j\alpha}$, $j \geq 0$.



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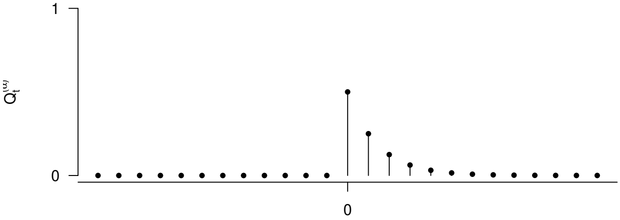
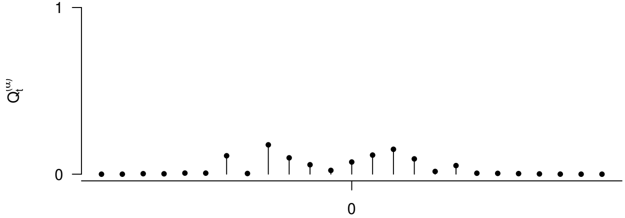


Causal solution to SRE under Kesten-Goldie assumptions

- (X_t) **causal solution to SRE**, $X_t = A_t X_{t-1} + B_t$, $((A_t, B_t))$ positive i.i.d. and $((A, B))$ satisfies Kesten-Goldie theory then

$$\Theta_t \quad = \quad A_t \cdots A_1 \,, \quad t \geq 0,$$

and $\Theta_t \to 0$ a.s. since $\mathbb{E}[\log(A_1)] < 0$ holds.



We take $A_t = e^{N_t - 1/2}$ such that (N_t) is i.i.d. gaussian noise, and we follow Example 6.1. in Janßen and Segers (2014) where $\Theta_{-t} = A_{-t} \cdots A_{-1}$, for $t \leq 0$.

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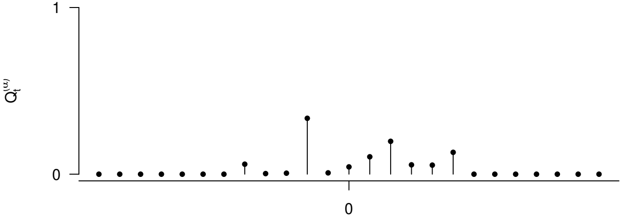

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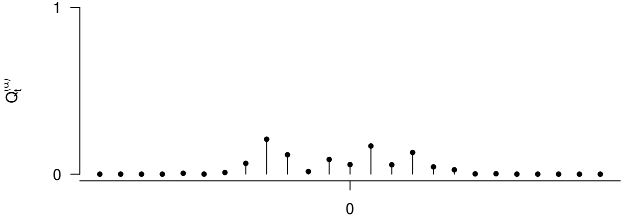

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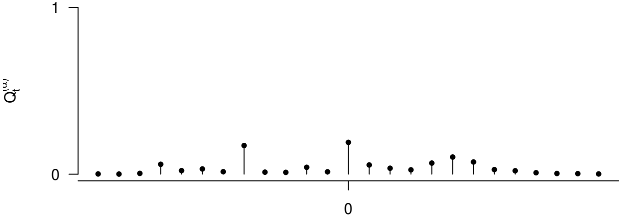

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Statistical methods

2- Present new statistical methodologies to **thoughtfully aggregate extreme recordings in space and time**.

- **Marginal tail inference of X_1.**
  1. G. Buriticá, P. Naveau. (2022). Stable sums to infer high return levels of multivariate rainfall time series. (Submitted). [arXiv]. (code)

- **Marginal tail inference of $\mathbf{X}_1$.**

  1. G. Buriticá, P. Naveau. (2022). Stable sums to infer high return levels of multivariate rainfall time series. (Submitted). [arXiv]. (code)

- **Inference clustering aspects of extremal ℓ^p-blocks**

  2. G. Buriticá, N. Meyer, T. Mikosch, O. Wintenberger. (2021). Some variations on the extremal index. Zap. Nauchn. Semin. POMI. Volume 501, Probability and Statistics. 30, 52—77. To be translated in J.Math.Sci. (Springer). [arXiv]. (code)
  3. G. Buriticá, T. Mikosch, O. Wintenberger. (2022) Large deviations of ℓ^p-blocks of regularly varying time series and applications to cluster inference. (Submitted). [arXiv].
  4. G. Buriticá, O. Wintenberger. Asymptotic normality for ℓ^p-cluster inference. (Ongoing).

p-CLUSTER INFERENCE

For inference purposes, let $b_n \to \infty$, $m_n = n/b_n \to \infty$.

$$\mathbf{X}_{[1,n]} = (\underbrace{\mathbf{X}_{[1,b_n]}}_{\mathcal{B}_{1,b_n}}, \underbrace{\mathbf{X}_{[b_n+1,2b_n]}}_{\mathcal{B}_{2,b_n}}, \dots, \underbrace{\mathbf{X}_{[n-b_n+1,n]}}_{\mathcal{B}_{m,b_n}}).$$

Aim:

Infer $\mathbb{E}[f(Y\mathbf{Q}^{(p)})]$ letting f act on extremal ℓ^p-blocks: $\|\mathcal{B}_{t,b_n}\|_p > x_{b_n}$, for suitable cluster functionals $f : \ell^p \to \mathbb{R}$, Y is (α)-Pareto distributed and independent of $\mathbf{Q}^{(p)}$.

We propose to estimate the statistic $f_p^{\mathbf{Q}} = \mathbb{E}[f(Y\mathbf{Q}^{(p)})]$ by

$$\widehat{f_p^{\mathbf{Q}}} := \frac{1}{k}\sum_{t=1}^{m} f\big(\mathcal{B}_t/\|\mathcal{B}_t\|_{p,(k+1)}\big)\mathbb{1}(\|\mathcal{B}_t\|_p > \|\mathcal{B}_t\|_{p,(k+1)}),$$

where $\|\mathcal{B}\|_{p,(1)} \geq \cdots \geq \|\mathcal{B}\|_{p,(m)}$.

Theorem[7] 3

Assume **AC**, $\mathbf{CS}_p$, and further mixing and bias conditions. There exists $k = k_n \to \infty$, $m/k \to \infty$, such that for suitable $f : \ell^p \to \mathbb{R}$,

$$\sqrt{k}(\widehat{f_p^{\mathbf{Q}}} - f_p^{\mathbf{Q}}) \xrightarrow{d} \mathcal{N}\big(0, \mathrm{Var}(f(Y\mathbf{Q}^{(p)}))\big),$$

[7] Similar arguments as in Kulik, Soulier (2020) and Cissokho, Kulik (2021)

1. We extend the inference to p-clusters, $p \le \infty$ selecting the blocks whose ℓ^p-norm exceed the high threshold x: $\|\mathcal{B}_t\|_p > x_{b_n}$.
2. We promote the use of order statistics of p-norm blocks such that

$$\|\mathcal{B}\|_{p,(k)}/x_{b_n} \xrightarrow{\mathbb{P}} 1.$$

   where $k_n = k_n(p) = \lceil m_n \mathbb{P}(\|\mathcal{B}_{1,b_n}\|_p > x_{b_n}) \rceil$. In this way k_n points to the bias-variance trade-off in extreme value statistics.
3. The same quantity $f_p^{\mathbf{Q}}$ can be estimated using different pairs p', $f_{p'}$ as

$$f_p^{\mathbf{Q}} = \mathbb{E}[f_p(Y\mathbf{Q}^{(p)})] = \frac{\mathbb{E}[\|\mathbf{Q}^{(p')}\|_p^\alpha f_p(Y\mathbf{Q}^{(p')}/\|\mathbf{Q}^{(p')}\|_p)]}{\mathbb{E}[\|\mathbf{Q}^{(p')}\|_p^\alpha]}.$$

4. p-cluster theory allows us to compare inference procedures:
   - Compare the suitable choices of $k_n = k_n(p) = \lceil m_n \mathbb{P}(\|\mathcal{B}_{1,b_n}\|_p > x_{b_n}) \rceil$.
   - Compare the asymptotic variances $\mathrm{Var}(f_p(Y\mathbf{Q}^{(p)}))$.

Denote $k_n(p) = \lceil m_n \mathbb{P}(\|\mathcal{B}_{1,b_n}\|_p > x_{b_n}) \rceil$ the extremal ℓ^p-blocks, for a sequence of levels (x_n) satisfying **AC**, $\mathbf{CS}_p$.

For i.i.d. sequence $k_n = \lceil n\mathbb{P}(|\mathbf{X}_0| > x_{b_n}) \rceil \sim k_n(\infty) \sim k_n(p) \sim k_n(\alpha)$ exceedances.

Heuristic on the number of extreme blocks:

$$k_n(p) \sim m_n \mathbb{P}(\|\mathcal{B}_1\|_p > x_{b_n}) \sim c(p) n \mathbb{P}(|\mathbf{X}_0| > x_{b_n}) \sim c(p) k_n \,,$$
$$k_n(\alpha) \sim m_n \mathbb{P}(\|\mathcal{B}_1\|_\alpha > x_{b_n}) \sim n \mathbb{P}(|\mathbf{X}_0| > x_{b_n}) \sim k_n \,,$$

thus $k_n(\infty) \lesssim k_n(p)$ for $p \in (\alpha, \infty)$ since $c(p) \downarrow c(\infty)$.

Assuming also $\mathbf{CS}_\alpha$, α-cluster inference is justified. In this case the tuning parameter k_n does not dependent on the underlying time dependencies.

Cluster-based extremal index inference

For example, if $f_\alpha : (\mathbf{x}_t) \mapsto \|(\mathbf{x}_t)\|_\infty^\alpha / \|(\mathbf{x}_t)\|_\alpha^\alpha$, then for $p = \alpha$,

$$\theta_{|\mathbf{X}|} \quad = \quad \mathbb{E}[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha] \quad = \quad c(\infty).$$

$\Longrightarrow$ New estimator of the extremal index based on extremal ℓ^α-blocks.

Examples

Letting $f_\alpha : \ell^\alpha \to \mathbb{R} : (\mathbf{x}_t) \mapsto \|\mathbf{x}\|_\infty^\alpha / \|\mathbf{x}\|_\alpha^\alpha$ act on extremal ℓ^α-blocks,

$$\mathbb{E}\big[\|\mathbf{Q}^{(\alpha)}\|_\infty^\alpha\big] \quad = \quad \theta_{|\mathbf{X}|}.$$

Letting $f_\infty : \ell^\infty \to \mathbb{R} : (\mathbf{x}_t) \mapsto \sum_{t\in\mathbb{Z}} \mathbb{1}(|\mathbf{x}_t| > 1)$ act on extremal ℓ^∞-blocks,

$$(\mathbb{E}[\textstyle\sum_{t\in\mathbb{Z}} \mathbb{1}(|Y\mathbf{Q}^{(\infty)}| > 1)])^{-1} \quad = \quad \theta_{|\mathbf{X}|}.$$

Numerical experiments

$$\widehat{\theta}_{|\mathbf{X}|,\alpha} = \frac{1}{k}\sum_{t=1}^{m} \frac{\|\mathcal{B}_t\|_\infty^\alpha}{\|\mathcal{B}_t\|_\alpha^\alpha} \mathbb{1}(\|\mathcal{B}_t\|_\alpha > \|\mathcal{B}\|_{\alpha,(k+1)}), \quad (1)$$

$$\widehat{\theta}_{|\mathbf{X}|,\infty} = \Big(\frac{1}{k}\sum_{t=1}^{n} \mathbb{1}(|\mathbf{X}_t| > \|\mathcal{B}_t\|_{\infty,(k+1)})\Big)^{-1}. \quad (2)$$

$$\widehat{\theta}_{|\mathbf{X}|,\alpha} = \frac{1}{k}\sum_{t=1}^{m} \frac{\|\mathcal{B}_t\|_\infty^\alpha}{\|\mathcal{B}_t\|_\alpha^\alpha} \mathbb{1}(\|\mathcal{B}_t\|_\alpha > \|\mathcal{B}\|_{\alpha,(k+1)}), \tag{1}$$

$$\widehat{\theta}_{|\mathbf{X}|,\infty} = \Big(\frac{1}{k}\sum_{t=1}^{n} \mathbb{1}(|\mathbf{X}_t| > \|\mathcal{B}_t\|_{\infty,(k+1)})\Big)^{-1}. \tag{2}$$

Consider a deterministic-threshold version of (2):

$$\widetilde{\theta}_{|\mathbf{X}|,\infty}(u) = \frac{\sum_{t=1}^{m} \mathbb{1}(\|\mathcal{B}_t\|_\infty > u)}{\sum_{t=1}^{n} \mathbb{1}(|\mathbf{X}_t| > u)}.$$

Then (2) is defined with $u = \|\mathcal{B}_t\|_{\infty,(k)}$. Instead, the so-called blocks estimator (Hsing, 1993) is defined letting $u = |\mathbf{X}_{(k)}|$.

$$\widehat{\theta}_{|\mathbf{X}|,\alpha} = \tfrac{1}{k}\textstyle\sum_{t=1}^{m} \frac{\|\mathcal{B}_t\|_\infty^\alpha}{\|\mathcal{B}_t\|_\alpha^\alpha}\, \mathbb{1}(\|\mathcal{B}_t\|_\alpha > \|\mathcal{B}\|_{\alpha,(k+1)}),$$
$$\widehat{\theta}_{|\mathbf{X}|,\infty} = \big(\tfrac{1}{k}\textstyle\sum_{t=1}^{n} \mathbb{1}(|\mathbf{X}_t| > \|\mathcal{B}_t\|_{\infty,(k+1)})\big)^{-1}.$$

- We simulate 1 000 AR(1) trajectories $(X_t)_{t=1,\dots,n}$, $X_t = \varphi X_{t-1} + Z_t$, for $n = 3\,000$.
- We fix $k = k_n = n/b_n^2$ and we use that $k_n(p) \sim m_n \mathbb{P}(\|\mathcal{B}_1\|_p > x_b) \sim n\, c(p)\mathbb{P}(|\mathbf{X}_0| > x_b) = o(n/b^{\alpha/p\vee 1})$.
- In this setting,

$$\mathrm{Var}(f_\alpha(YQ^{(\alpha)})) = \mathrm{Var}(f_\infty(YQ^{(\infty)})) = 0.$$

- The choice $p = \alpha$ is natural but requires the estimation of α. We use the estimator from de Haan *et al.* (2016).

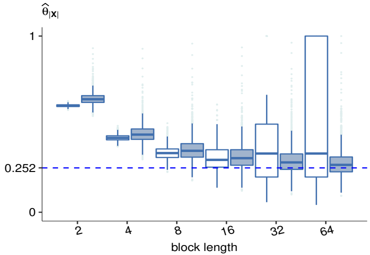


Figure 4: Boxplot of estimates $\widehat{\theta}_{|\mathbf{X}|,\widehat{\alpha}}$ (blue) and $\widehat{\theta}_{|\mathbf{X}|,\infty}$ (white), from observations $(\mathbf{X}_t)_{t=1,\dots,n}$ from a causal AR(1) model with student(α) noise, $\alpha = 1.3$ and $\varphi = 0.8$ (left), $\varphi = 0.5$ (right), such that $n = 3\,000$.

Simulation results

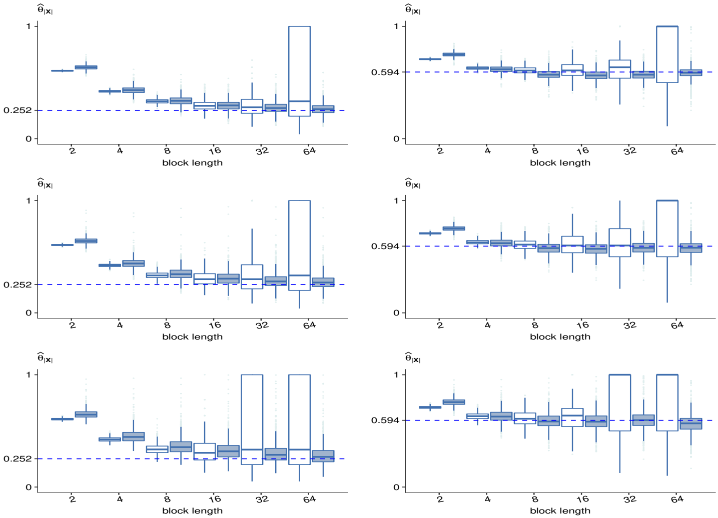


Figure 5: Boxplot of estimates $\widehat{\theta}_{|\mathbf{X}|,\widehat{\alpha}}$ (blue) and $\widehat{\theta}_{|\mathbf{X}|,\infty}$ (white). In the first row $n = 8\,000$, second $n = 4\,000$, third $n = 2\,000$.

Case study

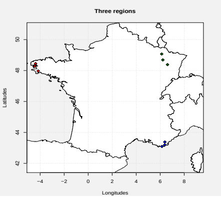
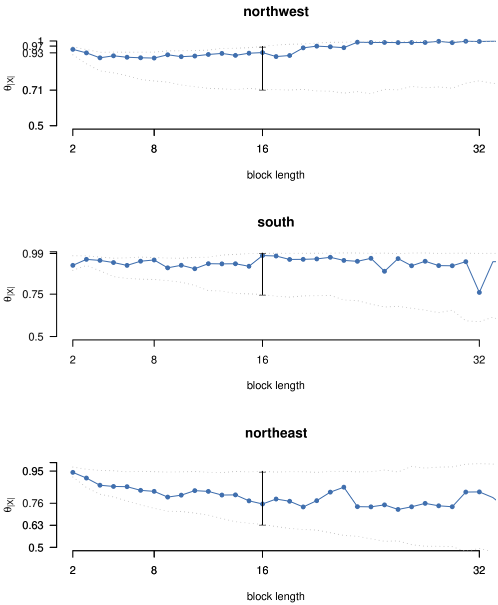


Figure 6: Location of weather stations in France



1. We presented large deviations of blocks in ℓ^p, i.e. the limit $\mathbf{X}_{[1,b_n]}/x_{b_n}$, such that $\|\mathbf{X}_{[1,b_n]}/x_{b_n}\|_p \xrightarrow{\mathbb{P}} 0$.
2. We developed the theory of p-clusters.
3. We studied p-cluster inference and compared inference procedures based on $p = \alpha$ and $p = \infty$.
4. We showed that the new inference methodologies using ℓ^p-blocks can be studied using p-cluster theory.
5. Inference based on extremal ℓ^α-blocks is robust to handle time dependencies but requires estimation of the (tail) inference α.
6. In the models we reviewed, the distribution of α-cluster is known. This could be used to propose new estimators of α-cluster statistics.

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Marginal tails inference

By Corollary 1,

$$\mathbb{P}(X_{0,j} > x_{b_n}) \quad \sim \quad m(j)\,(b_n)^{-1}\mathbb{P}(\|\mathcal{B}_{1,b_n}\|_\alpha > x_{b_n}), \tag{3}$$

where $m(j) = \mathbb{E}[(\Theta_{0,j})_+^\alpha] \in (0,1]$ traces the extremal spatial dependencies

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where $m(j) = \mathbb{E}[(\Theta_{0,j})_+^\alpha] \in (0, 1]$ traces the extremal spatial dependencies, and the distribution function

$$x \quad \mapsto \quad \mathbb{P}(\ \underbrace{\|\mathcal{B}_{1,b_n}\|_\alpha^\alpha}_{\sum_{t=1}^{b_n} |\mathbf{X}_t|^\alpha}\ \leq\ x\ ),$$

can be modeled from a stable distribution fitted to

$$\|\mathcal{B}_{1,b_n}\|_{\widehat{\alpha}}^{\widehat{\alpha}}\ ,\ \|\mathcal{B}_{2,b_n}\|_{\widehat{\alpha}}^{\widehat{\alpha}}\ ,\ \ldots\ ,\ \|\mathcal{B}_{m_n,b_n}\|_{\widehat{\alpha}}^{\widehat{\alpha}}.$$

By Corollary 1,

$$\mathbb{P}(X_{0,j} > x_{b_n}) \quad \sim \quad m(j)\,(b_n)^{-1}\mathbb{P}(\|\mathcal{B}_{1,b_n}\|_\alpha > x_{b_n}), \tag{3}$$

where $m(j) = \mathbb{E}[(\Theta_{0,j})_+^\alpha] \in (0,1]$ traces the extremal spatial dependencies, and the distribution function

$$x \quad \mapsto \quad \mathbb{P}(\underbrace{\|\mathcal{B}_{1,b_n}\|_\alpha^\alpha}_{\sum_{t=1}^{b_n}|\mathbf{X}_t|^\alpha} \;\le\; x\,),$$

can be modeled from a stable distribution fitted to

$$\|\mathcal{B}_{1,b_n}\|_{\widehat{\alpha}}^{\widehat{\alpha}}\,,\ \|\mathcal{B}_{2,b_n}\|_{\widehat{\alpha}}^{\widehat{\alpha}}\,,\ \ldots\,,\ \|\mathcal{B}_{m_n,b_n}\|_{\widehat{\alpha}}^{\widehat{\alpha}}.$$

Input: $\widehat{\alpha}^n, \widehat{m}^n(j), \mathbf{X}_1, \ldots, \mathbf{X}_n$.

Step 1: Find b such that $(\|\mathcal{B}_{t,b}\|_{\widehat{\alpha}^n}^{\widehat{\alpha}^n})_{t=1,\ldots,m}$ is nicely modeled by a stable distribution with unit stable parameter.

Step 2: Extrapolate high quantiles from stable distribution based on (3).

Step 3: Parametric bootstrap to obtain confidence intervals.

We run 1 000 trajectories with $n = 4\,000$ from four models with (tail) index 4. We use the estimator $\widehat{\alpha}^n$ from de Haan et al. (2016).

	Coverage probabilities for the $99.98th$ estimated quantile			
	Burr	Fréchet	Armax(0.7) [8]	Armax(0.8)
block maxima	.89	.93	.93	.91
peaks o. threshold	.85	.87	.79	.72
stable $b = 32$	**.94** (.51)	**.96** (.53)	.90 (.66)	.89 (.55)
stable $b = 64$	**.95** (.85)	.99 (.90)	**.96** (.89)	**.93** (.85)
stable $b = 128$	.98 (.94)	.99 (.91)	**.97** (.94)	**.95** (.92)

[8] The Armax(λ) model satisfies $X_t = \max\{\lambda X_{t-1}, (1-\lambda^\alpha)^{1/\alpha} Z_t\}$, $t \in \mathbb{Z}$, where (Z_t) are i.i.d. Fréchet distributed with (tail) index $\alpha > 0$.